Performance Comparison of the Run Sum *t*, EWMA *t* and Standard *t* Control Charts

Michael Boon Chong Khoo, Wai Chung Yeong, Jia Yu Choong, Wei Lin Teoh, and Sin Yin Teh

***Abstract*—**The *t*-chart is a rare event chart, where a rare event chart is a control chart that provides information about a process when the data comes from rarely occurring events. In this paper, a comparison of the performances of several *t* type control charts, like the run sum *t* chart, exponentially weighted moving average (EWMA) *t* chart and standard *t* chart are compared based on the average run length (ARL) criterion. The results show that among the *t* type charts considered, the run sum *t* chart gives smaller ARL values and is more sensitive than the EWMA *t* chart for detecting medium to large shifts in the process mean. The EWMA *t* chart is more sensitive than the run sum *t* chart for detecting small mean shifts. However, the standard *t* chart has the largest ARL compared with the run sum *t* chart and the EWMA *t* chart.

***Index Terms*—**Average run length, EWMA *t* chart, Run sum *t* chart, Standard *t* chart.

I. Introduction

The control chart is a very useful tool in determining if a manufacturing process is in a state of statistical control. A process is considered as operating in statistical control as long as the control chart does not show any out-of-control signal. $\bar{X}$ charts are often used to determine whether a process is in a state of statistical control. However, $\bar{X}$ charts do not operate effectively when the data comes from rarely occurring events. Hence, rare event charts are proposed.

A rare event chart is a chart that provides information about a process, where the data comes from rarely occurring events. Rare event charts were developed in response to the limitations of control charts in rare event scenarios. There are two types of rare event charts, which are the *g*- charts and *t*-charts.

A *t*-chart measures the time elapsed since the last event and creates a picture of a process over time. Each point on the chart denotes an amount of time that has passed since a prior occurrence of a rare event. A traditional plot of these data might contain many points at zero and an occasional point at one. A *t*-chart avoids flagging numerous points as out of control. The *t*-chart also helps to identify special and common causes of variations, so that appropriate improvements can be made.

Besides that, the *t*-chart is used to detect changes in the rate at which the adverse event occurs. When reading the *t*-chart, the points above the upper control limit (*UCL*) indicate that the amount of time between events has decreased. Thus, the rate of the events has increased.

This paper compares the *ARL* performances of the run sum *t* chart, EWMA *t* chart and standard *t* chart. The *ARL* is the average time for a control chart to issue an out-of-control signal. A control chart is better than its competitors if it has a smaller out-of-control *ARL* (ARL_1) for a fixed in-control *ARL* (ARL_0).

This paper is organized as follows. The next section reviews the *t* type charts. Subsequently, Section III compares the *ARL* performances of the *t* type charts. Finally, some concluding remarks are given in Section IV.

II. A Review On The T Type Control Charts

This section reviews the run sum *t* chart, EWMA *t* chart and standard *t* chart. In all the *t* charts reviewed, we let $\{X_{r,1}, X_{r,2}, ..., X_{r,n}\}$ be a sample at time *r* = 1, 2, …, where *n* is the sample size. Assume that there is independence within and between samples and $X_{r,s} \sim N\left(\mu_0 + \delta\sigma_0, b^2\sigma_0^2\right)$, for $1 \le s \le n$, where μ_0 and σ_0 are the nominal process mean and process standard deviation, respectively. The process is statistically in control if $\delta = 0$ and b = 1. Otherwise, the process has shifted when μ_0 has changed $(\delta \ne 0)$ or σ_0 has changed $(b \ne 1)$, or both μ_0 and σ_0 have changed.

The sample mean $\bar{X}_r$ and the sample standard deviation S_r at time *r* are defined as

$$\bar{X}_r = \frac{1}{n}\sum_{s=1}^{n} X_{r,s} \qquad (1)$$

and

$$S_r = \sqrt{\frac{1}{n-1}\sum_{s=1}^{n}\left(X_{r,s} - \bar{X}_r\right)^2}, \qquad (2)$$

respectively.

To improve the effectiveness against errors in estimating the process standard deviation, [1] proposed plotting the statistic T_r defined by

$$T_r = \frac{\bar{X}_r - \mu_0}{S_r/\sqrt{n}}\text{, for } r = 1, 2\ldots \qquad (3)$$

When the process is in-control (δ = 0 and *b* =1), the statistic

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T_r follows the Student's t distribution with $n-1$ degrees of freedom, while when the process is out-of-control, the statistic T_r follows a non-central Student's t distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta\sqrt{n}/b$.

A. The Run Sum t Chart

The run sum t chart was proposed by [2] and it was designed to enhance the sensitivity of the basic t chart while retaining the basic t chart's simplicity of implementation. The run sum t chart is divided into k regions above and k regions below the central line, CL. The k regions above the central line ($CL = 0$) of the chart are defined by k upper control limits (UCL) given by $0 = CL < UCL_1 < ... < UCL_{k-1} < UCL_k = \infty$ and k scores given by $0 \le S_1 \le S_2 \le ... \le S_k$, where each integer score S_j, for j =1, 2,…, k, is associated with the interval $\left[UCL_{j-1}, UCL_j\right)$. When j = 1, UCL_{j-1} = CL. The score function, S is defined as [2]

$$S\left(T_r\right) = S_j \text{ if } T_r \in \left[UCL_{j-1}, UCL_j\right), \text{ for } j = 1,2,\ldots,k. \quad (4)$$

The triggering score is $H = S_k$. H is set as S_k because the chart is designed to signal an out-of-control when a point T_r falls in the upper most region, i.e. $T_r \in \left[UCL_{k-1}, UCL_k\right)$.

Similarly, the k regions below the central line ($CL = 0$) of the chart are defined by the k lower control limits given by $-\infty = LCL_k < LCL_{k-1} < ... < LCL_1 < CL = 0$, where $LCL_j = -UCL_j$, for $j = 1,2,\ldots,k$. Also, the k scores are given by $-S_k \le -S_{k-1} \le ... \le -S_2 \le -S_1 \le 0$, where each integer score, $-S_j$, for j =1, 2, …, k, is associated with the interval $\left(LCL_j, LCL_{j-1}\right]$. When j =1, $LCL_{j-1} = CL$. The score function is defined as

$$S\left(T_r\right) = -S_j \text{ if } T_r \in \left(LCL_j, LCL_{j-1}\right], \text{ for } j = 1,2,\ldots,k. \quad (5)$$

The triggering score is $J = -S_k$. J is set equal to $-S_k$ as the chart is designed to signal an out-of-control when a point T_r falls in the lower most region, i.e. $T_r \in \left(LCL_k, LCL_{k-1}\right]$.

In this paper, the k regions above CL are denoted with scores $\{\alpha_1, \alpha_2, ..., \alpha_k\}$, while the k regions below CL are denoted with scores $\{-\alpha_1, -\alpha_2, ..., -\alpha_k\}$.

The regions below CL are symmetric to those regions above CL. The regions above CL and the regions below CL are used for detecting an increase and a decrease in the mean, respectively. The statistics monitored by the run sum t chart are based on the cumulative sums U_r and L_r which are defined as below:

$$U_r = \begin{cases} 0 & \text{if } T_r < CL \\ U_{r-1} + S\left(T_r\right) & \text{if } T_r \ge CL \end{cases}$$

and

$$L_r = \begin{cases} 0 & \text{if } T_r > CL \\ L_{r-1} + S\left(T_r\right) & \text{if } T_r \le CL \end{cases}$$

where r =1, 2,…, and $U_0 \ge 0$ and $L_0 \le 0$ are the initial values of the chart. U_r is the cumulative sum for T_r falling above CL, while L_r is the cumulative sum for T_r falling below CL. If $T_r < CL$, U_r is reset to zero. Similarly, if $T_r > CL$, L_r is reset to zero. An out-of-control signal is given at the rth sample if the cumulative score reaches or exceeds a positive critical score, $U_r \ge H$ or when it is less than a negative critical score, $L_r \le J$, where H and J are the triggering scores [2].

The upper control limits are defined as [2]

$$UCL_j = K \times F_t^{-1}\left(\alpha_j\right), \text{ for } j = 1, 2, \ldots, k-1. \quad (6)$$

Here, $F_t^{-1}(\ )$ is the inverse cumulative distribution function (cdf) of the Student's t distribution with $n-1$ degrees of freedom, while

$$\alpha_j = \Phi\left(\frac{3j}{k-1}\right), j = 1, 2, \ldots, k-1 \quad (7)$$

and $\Phi(\ )$ is the cdf of the standard normal distribution.

The optimization design of the run sum t chart involves finding the scores $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ and parameter K to minimize ARL_1, for the desired δ that needs a quick detection, based on the desired ARL_0 and n.

In this paper, an optimization program written in SAS, incorporating the Markov chain procedure proposed by [2] is used to compute the optimal parameters of the run sum t chart based on predefined values of ARL_0, n and δ. This paper considers the four regions run sum t chart ($k = 4$). The following steps explain how the ARL_1 values for the run sum t chart are computed:

Step 1: The desired ARL_0, n and δ values are specified in the program. The input parameters considered in this paper are ARL_0 = 370, $n \in \{3,5,10\}$, k = 4 and $\delta \in \{0.2, 0.6, 0.8, 1.0, 1.6, 2.0\}$.

Step 2: The optimal parameters K, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and the ARL value for a mean shift δ are computed by the SAS program using the Markov chain approach explained in [2].

B. The EWMA t Chart

The EWMA chart was first proposed by [3] and the EWMA t chart was proposed by [1]. The plotting statistic, Y_r for monitoring the process mean is given by [1]

$$Y_r = \lambda T_r + \left(1-\lambda\right)Y_{r-1}, \text{ for } r = 1, 2, \ldots \quad (8)$$

where T_r is given in (3) and $\lambda \in (0,1]$ is the smoothing constant, which is usually a small value, such as $\lambda = 0.1$ or $\lambda = 0.2$. When $\lambda = 1$, this chart simply becomes the t chart. Since the mean value of Y_r is equal to 0 when $\delta = 0$ and the t distribution is symmetrical about 0, $LCL = -UCL$. The UCL of the EWMA t chart is a function of λ, ARL_0 and n [1].

An out-of-control is issued if Y_r plots beyond the chart's limits. The Markov chain approach, originally proposed by [4], is used to evaluate the ARL_1 of the EWMA t control chart.

The optimization design of the EWMA t chart involves finding λ and UCL to minimize ARL_1, for a desired δ that needs a quick detection, based on the desired ARL_0 and n. In this paper, an optimization program written in SAS, incorporating the Markov chain procedure given in [5] is used to compute the optimal parameters of the EWMA *t* chart based on predefined values of ARL_0, *n* and δ. The following steps explain how the ARL_1 values for the EWMA *t* chart are computed:

Step 1: The desired ARL_0, *n* and δ values are specified in the program. The input parameters considered in this paper are ARL_0 = 370, $n \in \{3,5,10\}$ and $\delta \in \{0.2, 0.6, 0.8, 1.0, 1.6, 2.0\}$.

Step 2: The optimal parameters λ and *UCL*, as well as the ARL value for a mean shift δ are computed by the SAS program using the Markov chain approach given in [5].

C. The Standard t Chart

The standard *t* chart is constructed by plotting T_r in (3), based on the following control limits [1]

$$UCL = F_t^{-1}\left(1-\frac{\alpha}{2}\middle| n-1\right) \quad (9)$$

$$LCL = -UCL, \quad (10)$$

where $F_t^{-1}\left(.\middle| n-1\right)$ is the inverse cdf of the Student's *t* distribution with *n* – 1 degrees of freedom and α is the size of the Type-I error.

An out-of-control is signaled when T_r falls beyond the *LCL* or *UCL* limit. The *ARL* for the standard *t* chart is [1]

$$ARL = \frac{1}{1-F_t\left(UCL\middle| n-1,\delta\sqrt{n}/b\right)+F_t\left(LCL\middle| n-1,\delta\sqrt{n}/b\right)}, \quad (11)$$

where $1-F_t\left(.\middle| n-1,\delta\sqrt{n}/b\right)$ is the cdf of the non-central Student's *t* distribution with $n-1$ degrees of freedom and non-centrality parameter $\delta\sqrt{n}/b$.

In this paper, a program is written in SAS to compute the control limits *UCL* and *LCL*, as well as the *ARL* values of the standard *t* chart, based on predefined values of ARL_0, *n* and δ. The following steps explain how the ARL_1 values for the standard *t* chart are computed:

Step 1: The desired ARL_0, *n* and δ values are specified in the program. The input parameters considered in this paper are ARL_0 = 370, $n \in \{3,5,10\}$ and $\delta \in \{0.2, 0.6, 0.8, 1.0, 1.6, 2.0\}$.

Step 2: The control limits *UCL* and *LCL*, and the *ARL* value for a mean shift δ are computed by the SAS program using (9), (10) and (11), respectively.

III. Performance Comparison

A common way of comparing control charts is to compare their *ARL*s. *ARL* is used to approximately characterize the run length distribution. Charts having the same ARL_0 are compared in terms of their ARL_1 for various shifts in the process. Through this comparison technique, a chart is considered better in detecting a mean shift or process change if it has a smaller ARL_1 for the given shift size of interest. The *ARL* should be large when there is no change in the process but the *ARL* should be small when the process has undergone a change.

In this section, the ARL_1 performances of the run sum *t* chart, EWMA *t* chart and standard *t* chart are compared based on different combinations of the sample size (*n*) and size of mean shift (δ). The run sum *t* chart with four regions (*k* = 4) is adopted in this paper. The ARL_0 is fixed as 370. The effects of *n* and δ on the ARL_1 of the *t*-type charts are also studied.

Table I displays the results for the ARL_1 of the run sum *t*, EWMA *t* and standard *t* charts based on $n \in \{3,5,10\}$ and $\delta \in \{0.2, 0.6, 0.8, 1.0, 1.6, 2.0\}$.

TABLE 1: A COMPARISON OF ARL_1S OF THE RUN SUM *T*, EWMA *T* AND STANDARD *T* CHARTS, BASED ON $n \in \{3,5,10\}$ AND $\delta \in \{0.2, 0.6, 0.8, 1.0, 1.6, 2.0\}$

n	δ	Run sum *t*	EWMA *t*	Standard *t*
3	0.2	127.64	91.91	330.50
	0.6	18.45	22.62	178.39
	0.8	10.92	15.70	127.28
	1.0	7.62	11.94	93.06
	1.6	4.34	6.96	43.17
	2.0	3.59	5.50	29.00
5	0.2	78.98	39.42	267.78
	0.6	9.37	8.90	72.54
	0.8	5.82	6.12	40.08
	1.0	4.35	4.63	23.75
	1.6	2.56	2.70	7.08
	2.0	2.10	2.15	4.00
10	0.2	38.98	20.96	156.26
	0.6	4.57	4.36	15.85
	0.8	2.31	2.96	6.85
	1.0	3.12	2.23	3.58
	1.6	1.27	1.25	1.28
	2.0	1.05	1.05	1.05

From Table I, the run sum *t* chart outperforms the EWMA *t* chart for detecting medium and large mean shifts but the EWMA *t* chart outperforms the run sum *t* chart for detecting small mean shifts. For example, Table I shows that for *n* = 3, the ARL_1s of the run sum *t* chart when δ = 0.2, 0.6, 0.8, 1.0, 1.6 and 2.0 are 127.64, 18.45, 10.92, 7.62, 4.34 and 3.59, respectively. On the other hand, for *n* = 3, the ARL_1s of the EWMA *t* chart when δ = 0.2, 0.6, 0.8, 1.0, 1.6 and 2.0 are 91.91, 22.62, 15.70, 11.94, 6.96 and 5.50, respectively. For larger *n* and δ, the performance of the run sum *t* chart is similar with the EWMA *t* chart.

This shows that the run sum *t* chart is sensitive in detecting medium and large mean shifts, while the EWMA *t* chart is excellent in detecting small mean shifts. From Table I, the standard *t* chart is the worst in detecting all sizes of mean shifts.

Next, the performances of the run sum *t*, EWMA *t* and standard *t* charts as *n* and δ change are studied. From Table I, it is clearly seen that the ARL_1 decreases when δ increases. For instance, when *n* = 5, the ARL_1 of the run sum *t* chart decreases from 78.98 to 5.82 and then 2.56 when δ increases from 0.2 to 0.8 and 1.6, respectively. The results are the same for both the EWMA *t* and standard *t* charts. These results show that the greater the size of the mean shift, the smaller the ARL_1 which indicates that the control charts are more sensitive for detecting large shifts.

As *n* increases, the run sum *t*, EWMA *t* and standard *t* charts have better ARL_1 performance. Table I shows smaller ARL_1 values for *n* = 10, compared to *n* = 5 and *n* = 3 for the same δ. It is clearly seen that when *n* increases, the ARL_1 decreases. For instance, for $\delta = 0.2$, the ARL_1 of the run sum *t* chart decreases from 127.64 to 78.98 and 38.98 when n increases from 3 to 5 and then 10, respectively.

IV. CONCLUSION

This paper compares the performances of the t type control charts, like the run sum t, EWMA t and standard t charts based on their out-of-control ARLs (ARL_1s). Among the t type charts, the run sum t chart performs better than the EWMA t chart by giving a quicker out-of-control detection speed with smaller ARL_1s for medium and large shifts. However, the EWMA t chart outperforms the run sum t chart for detecting small shifts. For larger n and δ, the performance of the run sum t chart is similar with the EWMA t chart. On the other hand, the standard t chart which has the largest ARL_1 values among the t-type control charts is the worst in detecting errors in all sizes of mean shifts. The results of this comparison provide useful information to practitioners in deciding which chart to use in practice under different sizes of shift considerations.

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