Bifurcation Avoidance of Stable Periodic Motion in Non-Autonomous Dynamical System

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***Abstract*—**The author previously proposed a control methodology for the maximum local Lyapunov exponent (MLLE) of stable periodic motions existing in continuous-time non-autonomous dynamical systems. The MLLE of a stable periodic motion corresponds to the stability index of the motion. Since bifurcation of a stable periodic motion occurs when its stability changes by parameter change, this paper presents that the method proposed to control the MLLE of a stable periodic motion is available as a parametric controller to avoid occurrence of its bifurcation.

***Index Terms*—**Bifurcation Avoidance, maximum local Lyapunov exponent, periodic motion, non-autonomous dynamical system.

I. Introduction

Nonlinear difference and differential equations are widely used for mathematical modeling of physical systems [1], [2]. We generally set the values of system parameters to appropriate ones so that a desired behavior can appear in a steady state. However, a dynamical system may not work as expected for any reason, e.g., disappearance of the desired behavior for occurrence of bifurcation [3], [4].

From the fact that the maximum local Lyapunov exponent (MLLE) [5] of a stable periodic solution is related to their bifurcation, the author proposed a parametric controller on the MLLE of stable periodic points in discrete-time dynamical systems [6], [7]. The parameter regulation is derived from an optimization problem on the MLLE and a method of steepest descent. Moreover, using a stroboscopic mapping (otherwise known as Poincarè map) that transforms the trajectory of a continuous-time solution into a sequence of points, the author also proposed a control method of the MLLE for stable periodic motions in continuous-time non-autonomous dynamical systems [8]. This paper demonstrates that the proposed method [8] is available to avoid bifurcation of a stable periodic motion existing in Duffing equation.

II. Control Method of MLLE

I consider a continuous-time non-autonomous dynamical system defined by

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{p}) \tag{1}$$

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where t denotes the continuous time, $\boldsymbol{x}=(x_1, x_2, ..., x_N)^\top$ is the vector of state variables and $\boldsymbol{p}=(p_1, p_2, ..., p_M)^\top$ is the vectors of system parameters, where the superscript symbol $\top$ represents the transpose of a vector. Here, I assume that $\boldsymbol{f}$ is periodical on time and its period is Ψ.

I arrange local sections $\Omega=\{\boldsymbol{x}\in R^N \mid t=k\Psi, k=0, 1, 2, ...\}$ in (1) and express $\boldsymbol{x}(t)$ on the local section at $t=k\Psi$ as $\boldsymbol{x}^{(k)}$. Then, a stroboscopic mapping $\boldsymbol{S}$ can be defined as

$$\boldsymbol{S} : \boldsymbol{R}^N \to \boldsymbol{R}^N; \boldsymbol{x}^{(k)} \mapsto \boldsymbol{x}^{(k+1)}. \tag{2}$$

To simplify design of MLLE controller, I assume that $\boldsymbol{S}$ is differentiable with respect to t, $\boldsymbol{x}$, and $\boldsymbol{p}$ as many times as needed.

Using the stroboscopic mapping in (2), the derivative of $\boldsymbol{x}^{(k)}$ with respect to the initial point $\boldsymbol{x}^{(0)}$ can be defined as

$$\frac{\partial \boldsymbol{x}^{(k)}}{\partial \boldsymbol{x}^{(0)}} = D\boldsymbol{S}^k\left(\boldsymbol{x}^{(0)}, \boldsymbol{p}\right) = \prod_{\ell=1}^{k} D\boldsymbol{S}(\boldsymbol{x}^{(\ell)}, \boldsymbol{p}), \tag{3}$$

and the MLLE on $\boldsymbol{x}^{(k)}$ and $\boldsymbol{p}$ for the duration of $t\in[m\tau\Psi, (m+1)\tau\Psi]$ $(m=0, 1, 2, ...)$ can be defined as

$$\lambda^{(m\tau)} = \frac{1}{\tau}\sum_{k=m\tau}^{(m+1)\tau-1} \ln\left\| D\boldsymbol{S}(\boldsymbol{x}^{(k)}, \boldsymbol{p}) \cdot \boldsymbol{w}^{(k)} \right\|, \tag{4}$$

where $\boldsymbol{w}^{(k)}$ is a nearby point at the vicinity of $\boldsymbol{x}^{(k)}$, and $\|\cdot\|$ represents the Euclidean norm of a vector. The MLLE of a stable periodic solution to (1) is related to its stability. For example, if a target periodic solution is stable, then λ takes a negative value; when λ equals to zero, bifurcation of the stable periodic solution occurs.

To design a parametric controller so as not to make bifurcation for forcibly parameter change, I consider a minimization problem of an objective function defined by

$$J(\lambda^{(m\tau)}) = \frac{1}{2}\left(\lambda^{(m\tau)} - \lambda^*\right)^2 \tag{5}$$

where λ^* denotes the target value of the MLLE to be controlled and is set to a negative value by users.

Now, let p be a controlled parameter, which corresponds to one of $\boldsymbol{p}$, and I assume that $\lambda(m\tau)$ is differentiable with respect to p. Then, I obtain a gradient system of (5) with respect to p

defined by

$$p^{((m+1)\tau)} = p^{(m\tau)} - \eta \left(\lambda^{(m\tau)} - \lambda^* \right) \frac{\partial \lambda^{(m\tau)}}{\partial p} \quad (6)$$

where η is a user-defined positive parameter. Calculation of the derivative term at the right-hand side of (6) is described in [8].

III. Experimental Results

To verify whether the proposed controller defined in (6) is available to avoid bifurcation of stable periodic motions in continuous-time non-autonomous dynamical systems, I dealt with Duffing equation defined by

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -Cx_2 - x_1^3 + B_0(t) + B(t)\cos t \end{aligned} \quad (7)$$

where x_1 and x_2 are state variables, C is a constant parameter, B_0 and B are time-variant parameters, and the term "$B(t)$ cos t" corresponds to external input with period 2π.

Before numerical experiments, I analyzed sets of bifurcation points on some stable periodic motions using a method of bifurcation analysis [4]. Figure 1 plots bifurcation curves on a (B, B_0)-plane at C=0.2. The black solid curves indicated by I_n and G_n correspond to period-doubling and tangent (otherwise known as saddle node) bifurcation points, where the subscript number n is to distinguish different bifurcation curves.

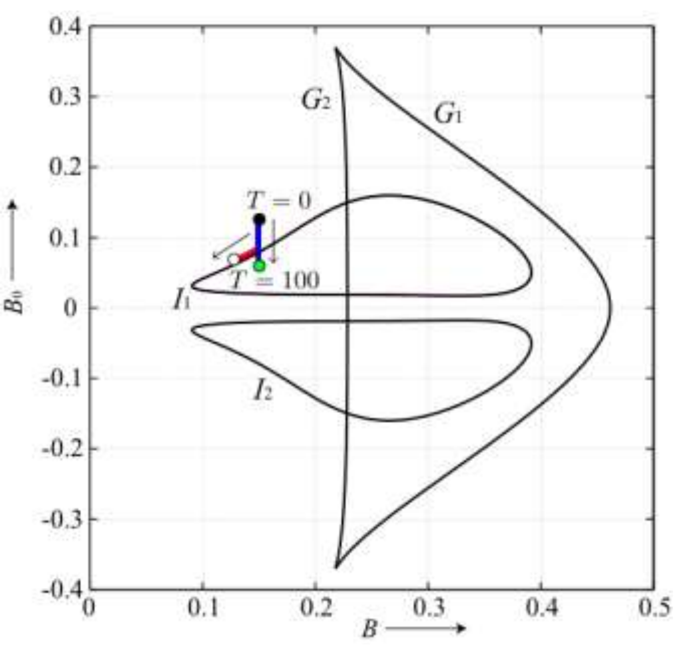

Fig. 1. Bifurcation points on periodic solutions to (1) on (B, B_0)-plain.

The MLLE of a stable periodic motion becomes almost zero at these bifurcation points. Because the stability of a periodic motion changes at the bifurcation points, we can observe another periodic motion with different aspect when a parameter value changes so as to pass through a bifurcation curve.

For example, we found a periodic motion with period 2π shown in Fig. 2 at the black point (C=0.2, B_0=0.12, and B=0.15); after the value of B_0 passed through the curve I_1 along the blue line, another periodic motion with period 4π shown in Fig. 3 was observed at the green point (C=0.2, B_0=0.07, and B=0.15), i.e., the period of the periodic motion became double. It is well known that the appearance of such irregular pulse with double period (alternating pulse) in the heart causes sudden cardiac death [9].

I applied the method proposed to control the MLLE to avoiding the period-doubling bifurcation caused by parameter change along the blue line. Here, I assume that the value of B is able to be handled; the value of B_0 is not able to be handled and is changed forcibly. I set the variation of $B_0(t)$ every interval of T=100Ψ to be −0.0005 so that it could cross the bifurcation curve I_1 along the blue locus starting from the black point. I also set to η=0.01 and λ^*=−0.1 in (6).

I showed experimental results in Figs. 1 and 4. The blue and red curves correspond to trajectories of the state variable and parameters without and with control. As you saw in Figs. 2 and 3, the stable one-periodic motion bifurcated on the curve I_1 without the proposed controller, and then, another stable two-periodic motion appeared. On the other hand, the red curve branching from the blue line in Fig. 1 represents the trajectory of the parameter values controlled by using the controller we designed. This red trajectory demonstrated that the proposed controller could be used to avoid the period-doubling bifurcation at which the period of the stable one-periodic motion becomes double.

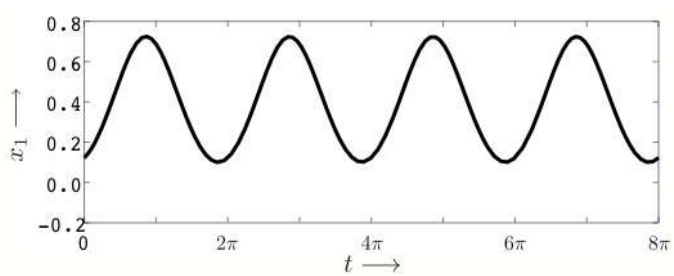

Fig. 2. Motion with period 2π at C=0.2, B_0=0.12, and B=0.15.

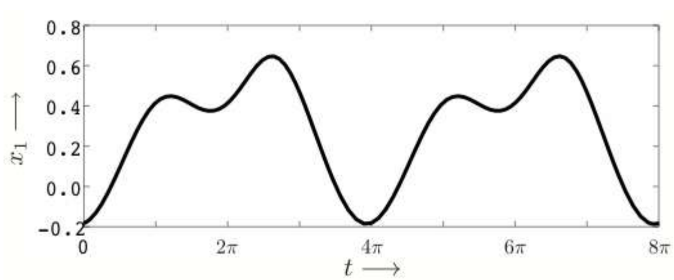

Fig. 3. Motion with period 4π at C=0.2, B_0=0.07, and B=0.15.

The detailed time series are shown in Fig. 4. The abscissa axes in the four graphs are the continuous time. The four graphs from the top to the bottom show the time evolution of x_1, λ, B, and B_0 where the blue and red sequences correspond to the trajectories both without and with control. I showed state variable x_1 as points on local sections arranged every Ψ, i.e., a stable one-periodic motion is shown a point and separated two points correspond to a stable two-periodic motion.

On the graph of x_1, we plotted 100 transient points on local sections at $t=mT$ (m=0, 1, 2, ..., 100). The blue points with $0 \leq t \leq 81T$ correspond to the one-periodic motion shown in Fig. 2. However, the value of x_1 in the blue points suddenly separated to two points (i.e., the two-periodic motion in Fig. 3 appeared) at $t = 82T$ without control because the parameter values passed through the period-doubling bifurcation point. This can be confirmed that the value of λ was almost zero at the time. In contrast, the red locus of x_1 indicates that the stable one-periodic motion in Fig. 5 remained for the duration of $82T \leq t \leq 100T$ because of the designed controller with adjusting the value of $B(t)$ after $t = 72T$ so that $\lambda = \lambda^*$.

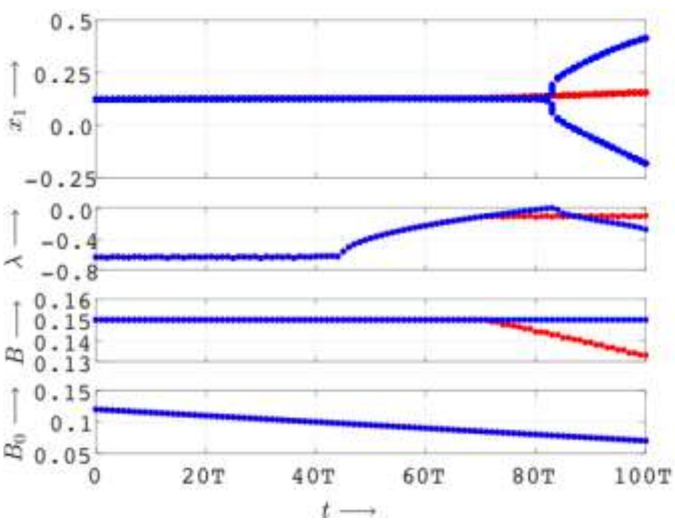

Fig. 4. Experimental results of bifurcation avoidance for a stable one-periodic motion.

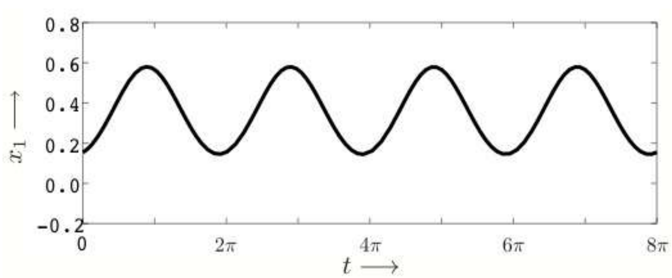

Fig. 5. Motion with period 2π at C=0.2, B_0=0.07, and B=0.133.

IV. Conclusion

In this paper, I considered the problem to avoid bifurcation of a stable periodic motion in Duffing equation. I also carried out experiments to verify whether the parametric controller proposed to control the MLLE can be used to avoid its bifurcation. The experimental results demonstrated that the proposed MLLE controller effectively worked to avoid a period-doubling bifurcation of a stable one-periodic motion.

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